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On the spectrum of the p-adic position operator

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Abstract. We propose a physical interpretation for a p-adic picture of reality. According to this interpretation, p-adic numbers describe measurements with a finite accuracy (at the same time, real numbers describe measurements with an infinite accuracy). We consider the representation of the position operator in a p-adic Hilbert space and study the spectrum of this operator.

1. Introduction

In the present paper we continue our investigations on a p-adic Hilbert space representation of quantum operators (see [1–5] for the first steps in this direction; see, for example, [6–9] for p-adic numbers and non-Archimedean analysis). We try to realize the following program. Consider the formal differential expression $\hat{\boldsymbol{H}} = H(\partial_{x_j}, x_j)$ of operators of quantum mechanics or quantum field theory. Let us realize this formal expression as a differential operator with variables x_j belonging to the field of p-adic numbers $\boldsymbol{Q}_p$ and study the properties of this operator in a p-adic Hilbert space. In [5] we have constructed a representation of the one-dimensional Weyl group in the L_2-space with respect to a p-adic Gaussian distribution [1, 4]. This representation differs radically from the 'ordinary one' in a complex Hilbert space (described by Weyl, von Neuman and others, see, for example, [10]). In particular, the operators of position $\hat{\boldsymbol{x}}$ and momentum $\hat{\boldsymbol{p}}$ are bounded in the p-adic L_2-space.

In the present paper we study the spectrum of the position operator. The main problem is to find the spectral set of this operator. This problem is sufficiently complicated and it is not yet solved completely. As $\hat{\boldsymbol{x}}$ is bounded, we know that its spectrum is a proper subset of $\boldsymbol{Q}_p$. It is contained in the ball of the radius $\lambda_x = \|\hat{\boldsymbol{x}}\|$. At the moment we cannot give the answer to the question: Does the spectrum of $\hat{\boldsymbol{x}}$ coincide with the ball of radius λ_x?

We have only proved that (in the case $p \neq 2$) the ball of radius $p^{-1/2(p-1)}\lambda_x$ belongs to the spectrum of $\hat{\boldsymbol{x}}$. The situation outside of this ball as well as the situation in the case $p = 2$ is not clear. Our proof is based on the point-wise properties of L_2-functions. Here the difference between the cases $p = 2$ and $p \neq 2$ is crucial. In the case $p \neq 2$ all L_2-functions are analytic on the ball of some radius which depends on the covariance of the p-adic Gaussian distribution.

Our investigations on spectral properties of the p-adic position operator gives us the possibility to present the following physical interpretation for p-adic quantum models: *p-adic numbers describe only measurements with a finite exactness.*

Therefore, the p-adic description extends essentially the quantum principle based on the Heisenberg-type uncertainty relations: *not only* might incompatible observables not be

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measured simultaneously with an infinite exactness, but also each singular observable might not be measured with an infinite exactness. The physical interpretation proposed in the present paper seems to be important for all of the p-adic mathematical physics [11–22] where the problem of interpretation has hindered further developments.

2. Numbers corresponding to the finite exactness of measurement

We always use real numbers to describe measurement procedures both in classical and quantum physics. This 'real description' has been used for a long time (at least since Galilei's work). Now practically nobody pays attention to one sufficiently strange aspect of the real formalism. Here we operate with physical quantities which might be measured with an infinite exactness. A real number has an infinite number of decimal digits and (at least theoretically) all these digits might be measured. However, every concrete experiment permits only a finite exactness of a measurement.

Is it possible to include this fixed exactness in a mathematical formalism? We shall try to do this.

What can we get in a measurement $\mathcal{S}$? Let us choose the unit of a measurement to be 1 and let us fix a natural number m (corresponding to the scale of this measurement). As results of $\mathcal{S}$ we can obtain only quantities of the form

$$x = \frac{x_{-k}}{m^k} + \cdots + \frac{x_{-1}}{m} + x_0 + \cdots + x_s m^s \tag{1}$$

where $x_j = 0, 1, \ldots, m-1$ are digits in our scale. Denote the set of all such x by $\boldsymbol{Q}_{m,fin}$.

Moreover, we could not approach an arbitrary finite exactness in $\mathcal{S}$. There exists a fixed number $k = k(\mathcal{S})$ such that the limit exactness of $\mathcal{S}$ is equal to $\delta(\mathcal{S}) = 1/m^k$. This means that we can be sure only in the digit x_{-k} but the next digit $x_{-(k+1)}$ is not well defined in $\mathcal{S}$ (in this fixed scale).

We wish to create a number system which describes only finite exactness of measurements. The set of 'physical numbers' $Q_{m,fin}$ will be taken to be the basis of our considerations.

First, we are interested in the construction of the field of real numbers R on the basis of $Q_{m,fin}$. The field R is the completion of $Q_{m,fin}$ with respect to the real metric $\rho(x, y) = |x - y|$ corresponding to the usual absolute value (valuation) $|\cdot|$. This metric describes absolute values of physical quantities (with respect to the fixed coordinate system). However, absolute values are not so important in quantum experiments. The exactness of a measurement is more important. We define on $Q_{m,fin}$ a new valuation corresponding to the exactness $\delta(\mathcal{S})$.

Set $|x|_m = m^k$ for x given by equation (1); (we assume that $x_{-k} \neq 0$).

It is a valuation:

- $|x|_m \geqslant 0$ and $|x|_m = 0$ iff $x = 0$;
- $|x y|_m \leqslant |x|_m |y|_m$;
- $|x + y|_m \leqslant \max(|x|_m, |y|_m)$ (the strong triangle inequality which implies the ordinary triangle inequality).

Set $\rho_m(x, y) = |x - y|_m$ and complete $\boldsymbol{Q}_{m,fin}$ with respect to this metric†. Denote this complete metric space by $\boldsymbol{Q}_m$ (m-adic numbers). This is a ring with respect to extensions of the usual operations of addition and multiplication. If $m = p$ is a prime number, then $\boldsymbol{Q}_p$ is a field (of p-adic numbers). The field of p-adic numbers is more well known in mathematics than the rings of m-adic numbers, see [6]. Of course, it would be better to

† This is the so-called ultrametric, i.e. the strong triangle inequality $r(x, y) \leqslant \max(r(x, z), r(z, y))$ holds.

work in a field than in a ring. However, from the physical point of view it is better to present the general scheme using m-adic numbers.

For any $x \in \boldsymbol{Q}_m$ we have a unique canonical expansion (converging in the $|\cdot|_m$-norm) of the form

$$x = x_{-n}/m^n + \cdots + x_0 + \cdots + x_k m^k + \cdots \tag{2}$$

where $x_j = 0, 1, \ldots, m-1$, are the 'digits' of the m-adic expansion. This expansion contains only a finite number of digits corresponding to negative powers of m. We interpret these numbers as providing a description of the finite exactness of a measurement. However, the expansion (2) shows that there is a new element in the m-adic description which is not present in the description of real numbers. There exist quantities described by (2) with an infinite number of digits corresponding to positive powers of m. It is very natural to consider such quantities as infinite quantities (with respect to our fixed unit 1). At the moment, we are not sure that such quantities might be useful in physics. However, there is always the possibility to restrict our attention to finite results.

Now the difference between real and m-adic descriptions of measurement is clear. If the exactness is infinite and the values of all observables are finite, then we have the real number description. If the exactness is finite and some values of observables may be infinite, then we have the m-adic number description.

Here m plays the role of a parameter characterizing the structure of the fixed scale. Different scales are useful for different experiments. However, different m-adic descriptions are (more or less) equivalent from the physical point of view. The exactness $\delta(\mathcal{S}) = 1/2^k$ (the 2-adic description) can be realized as $\delta'(\mathcal{S}) = 1/3^l$ (of course, not exactly but this suffices for applications). However, at the same time the rings $\boldsymbol{Q}_m$ and $\boldsymbol{Q}_{m'}, m \neq m'$, are not isomorphic. Thus, there is no mathematical equivalence of the descriptions.

As usual, we define balls in the metric space $\boldsymbol{Q}_m : U_r(a) = \{x \in \boldsymbol{Q}_m : \rho_m(x, a) \leqslant r\}, r = p^m, m = 0, \pm 1, \pm 2, \ldots$ and spheres $U_r(a) = \{x \in \boldsymbol{Q}_m : \rho_m(x, a) = r\}$. These sets are closed and open at the same time. Thus our Euclidean intuition does not work in this case. It is important to notice that the balls $U_r(0)$ are additive subgroups of $\boldsymbol{Q}_m$. Moreover, the ball $\mathbf{Z}_m = \mathcal{U}_1(0)$ is a ring, the ring of m-adic integers (canonical m-adic expansions of these numbers contain only non-negative powers of m).

Besides the strong triangle inequality we shall often use the following property of the p-adic valuation:

$$|a+b|_p = \max(|a|_p, |b|_p) \qquad \text{if } |a|_p \neq |b|_p. \tag{3}$$

We shall often use the equality

$$|n!|_p = p^{(n-S_n)/(1-p)} \tag{4}$$

where $n = \sum_j n_j p^j$ and $S_n = \sum_j n_j$ [6, 7] and we use the following lemma.

Lemma 2.1. $\quad S_{j+k} \leqslant S_j + S_k$.

The proof can be achieved by observing that the binomial coefficient $a!/b!(a-b)!$ is an integer and, therefore, its p-adic norm is less or equal to 1.

3. Pointwise properties of L_2-functions

Let b be a p-adic number, $b \neq 0$, the p-adic Gaussian distribution ν_b is defined as a Q_p-linear functional (on the space of polynomials) by its moments

$$M_{2n} = \int x^{2n} \nu_b(\mathrm{d}x) = (2n)! \frac{b^n}{n! 2^n} \qquad n \in \mathbb{N}_0$$

with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$;

$$M_{2n+1} = \int x^{2n+1} \ \nu_b(\mathrm{d}x) = 0 \qquad n \in \mathbb{N}_0.$$

We can extend the integration with respect to ν_b to some class of analytic functions from balls of Q_p to Q_p. Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \in Q_p$, be an analytic function on a ball U_τ, i.e. such that the terms $|a_k|_p \tau^k \to 0$, when $k \to \infty$. Then by definition

$$\int g(x)\nu_b(\mathrm{d}x) = \sum_{n=0}^{\infty} a_n M_n. \tag{5}$$

The following costant θ

$$\theta \equiv p^{1/2(1-p)} \sqrt{\frac{|b|_p}{|2|_p}}$$

will often be used in our paper.

Lemma 3.1. Let g be analytic on U_τ; if $\tau > \theta$ then the integral of equation (5) is convergent.

The proof is based on the estimate of the p-adic norm of the factorial.

Now let us introduce the analogue of Hermite polynomials on Q_p.

$$H_{n,b}(x) = \frac{n!}{b^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k} b^k}{k!(n-2k)!2^k}. \tag{6}$$

In the space $\mathcal{P}(Q_p)$ of Q_p-polynomials we introduce the inner product $(f, g) = \int f(x)\bar{g}(x)\nu_b(\mathrm{d}x)$. The polynomials $H_{n,b}$ verify the following othogonal conditions with respect to this inner product:

$$\int H_{m,b}(x) H_{n,b}(x)\nu_b(\mathrm{d}x) = \delta_{nm} n!/b^n. \tag{7}$$

Any $f \in \mathcal{P}(Q_p)$ can be written in the following way:

$$f = \sum f_n H_{n,b}(x). \tag{8}$$

We introduce the norm $\|f\|^2 = \max_n |f_n|_p^2 |n!|/|b|_p^n$ and we define $L_2(Q_p, \nu_b)$ as the completion of $\mathcal{P}(Q_p)$ with respect to $\| \ . \ \|$.

Theorem 3.1. The space $L_2(Q_p, \nu_b)$ is the set

$$\left\{ f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x),\ f_n \in Q_p :\ \text{the series } \sum_{n=0}^{\infty} |f_n|^2 n!/b^n \text{ converge in } Q_p \right\}.$$

For $f \in L_2(\boldsymbol{Q}_p, \nu_b)$ we set

$$\sigma_n^2(f) = |f_n|_p^2 |n!/b^n|_p$$

where f_n are the Hermite coefficients of f given by the expression

$$f_n = \frac{b^n}{n!} \int f(x) H_{n,b}(x) \nu_{b,p}(\mathrm{d}x). \tag{9}$$

We formally define the following map,

$$\Delta(\lambda, x) = \sum_{m=0}^{\infty} \frac{b^m}{m!} H_{m,b}(\lambda) H_{m,b}(x) \nu_b(\mathrm{d}x)$$

which formally verifies the following relation for any map $f \in L_2(Q_p, \nu_b)$:

$$\int \Delta(\lambda, x) f(x) \nu_b(\mathrm{d}x) = f(\lambda).$$

Now we wish to study the relations between $L_2(Q_p, \nu_b)$ functions and analytic functions.

Theorem 3.2. Assume $p \neq 2$. Let $f \in L_2(Q_p, \nu_b)$. Then f is analytic in each ball of radius $\tau < \theta$. Conversely, if f is analytic in U_τ, and $\tau > \theta$, then $f \in L_2(Q_p, \nu_b)$.

Proof. If $f \in L_2(Q_p, \nu_b)$, then, by using the explicit expression of the $H_{n,b}(x)$ polynomials, we have

$$f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x) = \sum_{j=0}^{\infty} c_j x^j$$

where the coefficients c_j are

$$c_j = \frac{1}{b^j j!} \sum_{k=0}^{\infty} \frac{f_{j+2k}(j+2k)!}{(-2b)^{2k} k!}.$$

Now, since

$$|c_j|_p \leqslant \frac{1}{|b|_p} \frac{1}{|n!|_p} \max_k |f_{j+2k}|_p \sqrt{\frac{|(j+2k)!|_P}{|b|_p^{j+2k}}} \sqrt{|(j+2k)!|_p} \frac{|b|_p^{j/2}}{|k!|_p |2|_p^k}$$

and the fact that $f \in L_2(Q_p, \nu_b)$, we get

$$|c_j|_p \leqslant \|f\| \frac{1}{|b|_p} \frac{1}{|n!|_p} \max_k \sqrt{|(j+2k)!|_p} \frac{|b|_p^{j/2}}{k! |2|_p^k}.$$

By using equation (4), we can show that

$$\|c_j|_p \leqslant |f|_p \frac{1}{(|b|_p)^{j/2}} p^{j/2(p-1)}$$

which, in turn, completes the proof. □

Conversely, if f is analytic in U_τ, we formally compute the cofficients of f written on Hermitian polinomials, which are

$$f_n = \frac{b^n}{n!} (-1)^n \sum_{m=0}^{\infty} c_{n+2m} b^m \frac{(n+2m)!}{2^m m!}$$

which satisfy the requirement

$$\sigma_m^2(f) \to 0 \qquad \text{when } n \to \infty$$

when $\tau > \theta$. It is possible that our estimates are not the best ones.

Now we wish to discuss the possibility to extend the integral to $L_2(Q_p, \nu_b)$ functions. Since every function $f \in L_2(Q_p, \nu_b)$ is analytic in U_τ, when $\tau < \theta$ we can define

$$\int f(x) \nu_b(\mathrm{d}x) = \int \sum_j c_j x^j \nu_b(\mathrm{d}x) \tag{10}$$

by re-arranging the series where it is convergent. On the other hand, if we define the integral of $f \in L_2(Q_p, \nu_b)$ by

$$\int f(x) \nu_b(\mathrm{d}x) = \sum_{n=0}^{\infty} \int f_n H_{n,b}(x) \nu_b(\mathrm{d}x) \tag{11}$$

by using the orthogonal relations between the Hermitian polynomials we obtain

$$\int f(x)\nu_b(\mathrm{d}x) = f_0.$$

Theorem 3.3. Let $f \in L_2(Q_p, \nu_b)$ be analytic on the ball U_τ; then, the two definitions of the integrals given by equations (10) and (11) coincide.

4. On the spectral set of the position operator

Using the pointwise properties of L_2-functions, we may prove that the set of λ, $|\lambda|_p < \theta$, belongs to the spectrum of the position operator (in the case $p \neq 2$). At the moment, it is not clear whether this set is a proper subset of the spectrum or the whole spectrum.

Theorem 4.1. Let $p \neq 2$ and let $|\lambda|_p < \theta$. Then λ belongs to the spectrum of the position operator $\hat{\boldsymbol{x}}$.

Proof. Consider the equation $(\hat{\boldsymbol{x}} - \lambda) f(x) = 1$ in the space $L_2(Q_p, \nu_b)$. By theorem 3.2 we have that f is analytic on the ball of radius τ where $\tau < \theta$. We can choose $\tau > |\lambda|$. Thus, this equation can be considered as the equation for analytic functions on the ball $\mathcal{U}_\tau$. Of course, this equation has no solution in the space of analytic functions and consequently in the standard L_2-space. □

In quantum mechanics over the real numbers, the point spectrum of the position operator $\hat{\boldsymbol{x}}$ is empty, i.e. $\hat{\boldsymbol{x}}$ has no eigenvalues $\lambda \in \mathbb{R}$. If we consider the standard representation of $\hat{\boldsymbol{x}}$ in the space $L_2(\mathbb{R}, \mathrm{d}x)$, then the equation

$$\hat{\boldsymbol{x}} f_\lambda(x) = \lambda f_\lambda(x) \tag{12}$$

has no solutions in $L_2(\mathbb{R}, \mathrm{d}x)$ for any $\lambda \in \mathbb{R}$. Here, in fact, $f_\lambda(x) = \delta(x - \lambda)$, but the δ-function does not belong to $L_2(\mathbb{R}, \mathrm{d}x)$.

If we change the representation and realize $\hat{\boldsymbol{x}}$ as the multiplication operator in the space $L_2(\mathbb{R}, \nu_b)$, the situation does not change essentially. The equation (12) does not have any solution $f_\lambda \in L_2(\mathbb{R}, \nu_b)$.

In this section we shall show that the p-adic picture does not differ from the real one.

Theorem 4.2. Let $p \neq 2$. Then the point spectrum of the position operator $\hat{\boldsymbol{x}} : L_2(\boldsymbol{Q}_p, \nu_b) \to L_2(\boldsymbol{Q}_p, \nu_b)$ is the empty set.

The proof of this theorem is rather long. We divide it into a few steps. As usual in p-adic analysis, the case $p = 2$ is exceptional. There we shall prove that $\hat{\boldsymbol{x}}$ has no eigenvalues outside of the sphere $S_\theta(0)$. On this sphere our proof does not work.

Lemma 4.1. Suppose that the equation (12) has a non-zero solution f_λ belonging to $L_2(\boldsymbol{Q}_p, \nu_b)$. Then we have the following formula for the Hermite coefficients $f_{\lambda,n}$ of the function f_λ :

$$f_{\lambda,n} = \frac{b^n}{n!} H_{n,b}(\lambda) = \sum_{k=0}^{[n/2]} \frac{(-1)^k \lambda^{n-2k} b^k}{k!(n-2k)!2^k} \tag{13}$$

where we have choosen the normalization constant $c = \int f_\lambda(x)\nu_{b,p}(\mathrm{d}x) = 1$.

Proof. Using equation (12), we get $\int x^k f_\lambda(x)\nu_{b,p}(\mathrm{d}x) = \lambda^k$. To obtain formula (13), it suffices to use the general formula (9) for Hermite coefficients. □

Proposition 4.1. The point $\lambda = 0$ does not belong to the point spectrum of the position operator $\hat{\boldsymbol{x}}$.

Proof. Using (13), we have $f_{0,2m+1} = 0$ and $f_{0,2m} = (-1)^m b^m/2^m m!$, $m = 0, 1, \dots$. We obtain

$$\sigma_{2m}^2(f_0) = |2|_p^{-2m}|2m!/m!^2|_p = |2|_p^{-2m} p^{(S_{2m}-2S_m)/p-1}.$$

Now we show that there exists a subsequence $\{m_k\}_{k=0}^\infty$ such that σ_{2m_k} does not approach to zero for $k \to \infty$. It suffices namely to choose $m_k = p^k$. Here if $p \neq 2$, then $S_{2p^k} = 2$, i.e. $S_{2p^k} - 2S_{p^k} = 0$. Thus, $\sigma_{2p^k}^2(f_0) = 1$ for all k. If $p = 2$, then $S_{2^{k+1}} = S_{2^k}$, i.e $S_{2^{k+1}} - 2S_{2^k} = -1$. Thus, $\sigma_{2^{k+1}}^2(f_0) = 2^{2^{k+1}-1}$. □

Furthermore, we shall prove that for small $|\lambda|_p$, the behaviour of the Hermite coefficients $f_{\lambda,2p^k}$ coincides with the behaviour of $f_{0,2p^k}$.

Lemma 4.2. Let $|\lambda|_p < \theta$, then

$$|f_{\lambda,2p^k}|_p = |f_{0,2p^k}|_p. \tag{14}$$

Proof. We shall use the property (3) of the p-adic valuation. We rewrite the expression (13) for the Hermite coefficients in the form

$$f_{\lambda,2m} = (-1)^m \sum_{j=0}^m \frac{(-1)^j\lambda^{2j}b^{m-j}}{(m-j)!(2j)!2^{m-j}} = \sum_{j=0}^m a_j.$$

Here $a_0 = f_{0,2m}$. Furthermore, we rewrite a_j, $j = 1, \dots, m$, in the form

$$a_j = a_0(-1)^j\left(\frac{2\lambda^2}{b}\right)^j \frac{m!}{(m-j)!(2j)!}.$$

By (4) we get

$$|m!/(m-j)!(2j)!|_p = p^{j/(p-1)}p^{S(m,j)/(p-1)}$$

where $S(m, j) = S_m - S_{m-j} - S_{2j}$. Finally, we have

$$|a_j|_p = |a_0|(|\lambda|/\theta)^{2j}p^{S(m,j)/(p-1)}.$$

We always have $S_{m-j} + S_{2j} \geqslant 1$. If $m = p^k$, then $S_m = 1$. Hence, in this case $S(m, j) \leqslant 0$. Thus, $|a_j|_p < |a_0|_p$ for all $j = 1, \dots, p^k$. □

Lemma 4.3. If $p \neq 2$, then the equality (14) is also valid on the sphere $S_\theta(0)$.

Proof. It suffices to show that $S(m, j) < 0$ for all $j = 1, \dots, m = p^k$. If $j \neq m$, then $S_{2j} \geqslant 1$ and $S_{m-j} \geqslant 1$. If $j = m$, then $S_{m-j} = 0$, but $S_{2j} \geqslant 2$ (because if $S_{2j} = 1$, then $2j = p^l$). □

Remark. This proof does not work in the case $p = 2$. Here $S_{2^{k+1}} = S_{2^k}$.

Heuristically it is evident that the term with the maximal power of λ in (13), namely

$$a_m = \frac{\lambda^{2m}}{2m!} \tag{15}$$

has to dominate for respectively large $|\lambda|_p$. First of all, we study the L_2-behaviour of these coefficients.

Lemma 4.4. The function $g_\lambda(x) = \sum_{m=0}^{\infty} a_m H_{2m,b}(x)$ does not belong to the space $L_2(\boldsymbol{Q}_p, \nu_b)$ for all λ satisfying the inequality $|\lambda|_p \geqslant \theta$.

Proof. By (4) we have

$$\sigma_{2m}^2(g_\lambda) = |\lambda|_p^{4m}/|b|_p^{2m}|2m!|_p = (|\lambda|_p/\theta)^{4m}|2|_p^{-2m}p^{-S_{2m}/(p-1)}.$$

Set $m = p^k$, then $S_{2p^k} = 2$ for $p \neq 2$ and $S_{2^{k+1}} = 1$ for $p = 2$. In any case $\sigma_{2p^k}(g_\lambda) \not\to 0, k \to \infty$, if $|\lambda|_p \geqslant \theta$. □

Lemma 4.5. If $|\lambda|_p > \theta$, then $|f_{\lambda,2p^k}|_p = |a_m|_p$ for sufficiently large k.

Proof. It is more convenient to rewrite the Hermite coefficients in the form

$$f_{\lambda,2m} = \sum_{k=0}^{m} \frac{(-1)^k \lambda^{2m-2k} b^k}{k!(2m-2k)!2^k} = \sum_{k=0}^{m} a_{m-k}.$$

We show that the term a_m strictly dominates in this sum. Let $k = 1, \ldots, m$. We have

$$|a_m|_p = |\lambda^{2m}/2m!|_p|b^k/2^k\lambda^{2k}|_p|2m!/k!(2m-2k)!|_p.$$

By (4) we get $|2m!/k!(2m-2k)!|_p = p^{-k/(p-1)}p^{A(m,k)/(p-1)}$, where $A(m,k) = S_{2m} - S_k - S_{2(m-k)}$. Hence we get $|a_{m-k}|_p = |a_m|_p(\theta/|\lambda|_p)^{2k}p^{A(m,k)/(p-1)}$.

Now set $m = p^l$. Consider the case $p \neq 2$. Here $S_{2p^l} = 2$. If $m \neq k$, then $S_k + S_{2(m-k)} \geqslant 2$ and, consequently, $A(m,k) \leqslant 0$. Now let $m = k$, then $S_{2(m-k)} = 0$ and $S_{2m} - S_m = 1$. Hence, $p^{A(m,k)/(p-1)} = p^{1/(p-1)}$. Thus, we have $|a_{m-k}|_p = |a_m|(\theta/|\lambda|_p)^{2k}$ for $k = 1, \ldots, m-1$, and $|a_0|_p = |a_m|_p(\theta/|\lambda|_p)^{2m}p^{1/(p-1)}$. As $|\lambda|_p > \theta$, both these quantities are less then $|a_m|_p$ for sufficiently large $m = p^l$. Now consider the case $p = 2$. Here $S_{2m} = S_m = 1$. If $k \neq m$, then $S_k + S_{2(m-k)} \geqslant 2$. If $k = m$, then $A(m,k) = 0$. □

Proof of theorem 4.2. If $|\lambda|_p \leqslant \theta$, then the term $a_0 = f_{2m,0}$ dominates and we have $|f_{\lambda,2m}|_p = |f_{0,2m}$ for $m = p^k$. We need only to use proposition 4.1. If $|\lambda|_p > \theta$, then the term a_n dominates and $|f_{\lambda,2m} = |a_m|_p$ for $m = p^k$. Thus, we need only to use the last lemma to conclude. □

In the same way we prove

Theorem 4.3. Let $p = 2$ and $\lambda \notin S_\theta(0)$, then λ is not an eigenvalue of the position operator $\hat{\boldsymbol{x}}$.

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